GEOGEBRA TO HELP IN THE UNDERSTANDING AND MEMORIZING MATHEMATICAL FORMULAS

Mr Miroslava Mihajlov Carević[1]; Dr Nebojša Denić[2]
[1] Faculty of Mathematics and Computer Science, Alfa BK University, Beograd, SERBIA, miroslava.carevic.mihajlov@alfa.edu.rs
[2] Faculty of Mathematics and Computer Science, Alfa BK University, Beograd, SERBIA, nebojsa.denic@alfa.edu.rs

Apstrakt: *In this paper we deal with examining the efficiency of the visual display of mathematical regularities in the process of learning algebraic formula.Visualization of the display is made by computer support and GeoGebra. After checking the knowledge of students in the seventh grade knowledge of basic mathematical formula, we concluded that students partially governed this matter. We decided to commit two additional time for mastering the formula.In order to compare the results we formed the experimental and control group. In the experimental group students worked in cooperative groups with computers and GeoGebra while in the control worked with the teachers and exercise on the board. For the third time we again tested the students. The results of the experimental group were significantly better than the control. After three weeks we repeated the test. The difference in performancet was then even higher. It is shown that the visual representation of certain laws contributes to its understanding and long-term memory.*

Keywords*: visualization, GeoGebra, computer support.*

1. INTRODUCTION

During school, students meet a large number of formulas both in mathematics and in the other teaching fields. For most students, understanding and remembering the formula is a big problem. Geometric formulas and laws, by their very nature, have a visual interpretation, so the pupils have a noticeably higher linkage of materials and the independent execution of formulas as well as better application in tasks. In algebraic formulas, students encounter difficulties because they are forced to memorize them in a derived, final form. Algebraic formulas are derived, proven and applied in textbooks and mathematics teaching, but their visualization is not shown, which directs students to memorizing the formula without understanding the lawfulness expressed in this formula.
Representation and visualization are of paramount importance in the process of learning and understanding mathematics (Duval [1], 1999). They are also extremely useful, and sometimes necessary in solving mathematical tasks. Visualization as a means in understanding, interpreting mathematical principles and as a means to solve mathematical problems becomes more and more present in mathematical education (Arcavi[2], 2003). By acquiring skills for understanding structural characteristics and solving problems in mathematics using visualization and animation, were concerned other researchers in mathematical education (Scheiter, Gerjets & Schuh[3], 2010; Rivera & Becker [4], 2008; Van Garderen & Montague[5], 2003; Pape, Bell & Yetkin [6], 2003).
The problem of research in this paper is the efficiency of learning and understanding mathematical formulas using modern technologies in education. The research was carried out at elementary school "Stevan Sremac" in Belgrade during the school year 2013/14. After checking the knowledge of students of the seventh grade of elementary school in the knowledge of basic mathematical formulas, we concluded that the students partially rule this matter. In order to acquire greater and more reliable knowledge of students, we decided to dedicate two additional hours to mastering mathematical formulas covered by previous knowledge test. We have formed an experimental and control group of students in which we have realized the intended content in different ways. In the experimental group, we organized a pictorial representation of the laws expressed in mathematical formulas, on computers via GeoGebra. In this group, we also organized collaborative learning in three-member groups recommended by education professionals as one of the most advanced tools for improving teaching and learning (Gomez & Passerini[7], 2010; Garcia[8], 2013). The students in the control group worked in the classroom practicing and applying the formulas in various tasks with the help of teachers. Both groups, experimental and control, are composed of two classes selected so that the average grade in mathematics is approximately the same as the number of students in the groups. The procedure and description of the research are presented in Section 3. Upon completion of the research, a check was made of the acquired knowledge of the students of both groups. The obtained results are shown, analyzed and compared to Section 4.

2. NEW TEHNOLOGIES IN TEACHING MATHEMATICS

New technological advancements have significantly increased the range of teaching resources in education and also set new demands on teachers as teaching staff (Ruthven[9], 2009). Therefore, it is necessary to develop adequate pedagogical methods for the use of modern technologies in education (Manenova, Skutil & Zikl[10], 2010; Valtonen et al. [11], 2014; Viamonte[12], 2010). Over the past decades, there has been the development of mathematical software packages, GeoGebra, Geometr's Sketcpad, Cabri Geometry. A large number of scientific studies include examining the effectiveness of mathematics learning using software mathematical packages (Takači, Stankov & Milanović[13], 2015; Ruthven, Hennessy & Deaney[14], 2008; Laborde[15], 2001). Also, testing the performance of GeoGebra and visualization in mathematics teaching in the IWB (Interactiv White Board) equipped classrooms (Lavicza & Papp-Varga[16], 2010; Walny et al. [17], 2011).
Modern technologies require technologically trained, educated members of society. Therefore, one of the main goals of education is to train students to use modern technologies actively and purposefully.

3. METODOLOGY OF RESEARCH

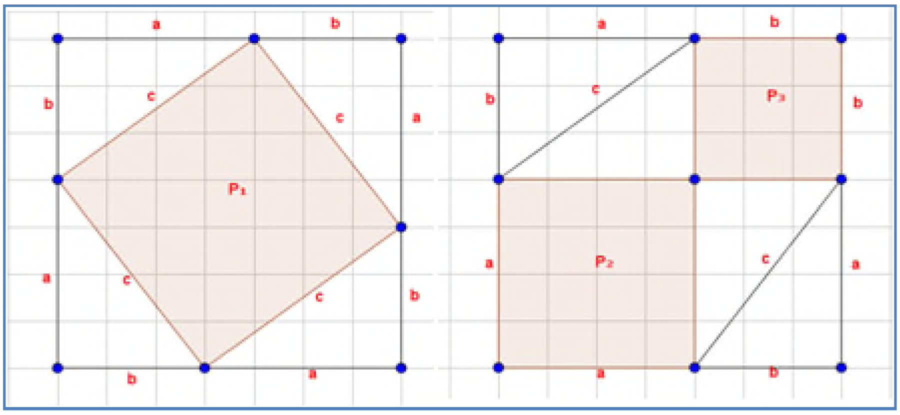
Getting to know students with Pythagoras's theorem begins in the first semester of the seventh grade. The processing and application of the theorem in a straight triangle is planned and then in rectangle, square, equilateral triangle, rhombus and trapeze. Students who have mastered the theorem are easily applied in every right triangle and in straight triangles, which are integral parts of other geometric figures, thus reaching the formula for calculating the elements of the aforementioned geometric figures. Students who did not master the theorem are not able to follow the procedure of applying the theorem in other geometric figures, so they are forced to learn the formulas used to calculate the elements of other geometric figures. It's a painstaking, often futile job and certainly not productive. Students should be trained to use acquire knowledge in achievement new knowledge, but it is necessary to master the basic knowledge previously.
After processing the Pythagorean theorem and its applications in geometric figures, students of the seventh grade are familiar with algebraic expressions and algebraic formulas for the square of the binomial and the difference of squares. Following the accomplishments of students during the realization of this matter, we realized that most students identify the difference in squares, $a^2 – b^2$, with the square of the difference, more precisely with the square of the binomial $(a–b)^2$. For a large number of students $(a + b)^2$ equivalent to $a^2 + b^2$.
In order to gain a more precise and complete insight into the students' knowledge of this matter as well as the Pythagorean theorems, we carried out student testing. On this pre-test, five tasks were assigned to check the knowledge of elementary facts. In the first and second task, it was required to calculate the unknown page of a straight triangle, where in the first task the unknown page was hypotenuse and in the second catheter. In the third task it was checked the knowledge of the formula for calculating the squares of a collection $(a + b)^2$, in the fourth square of the difference $(a–b)^2$ and in the fifth difference square $a^2 – b^2$. Every exact task brought one (1) point to the students. Half-finished tasks are not scored. Pre-test results are shown in Table 1.
The obtained results showed that the students did not learn how to apply the predicted formulas. Of the total of 130 points on the test (26 students per maximum 5 points), students in 7_1 scored 51 (or 39,2%). In 7_2 of the possible 120 points, the students achieved a total of 42 (or 35%). The students in 7_3 from the possible 125 points scored 44 (or 35,2%) while the students in 7_4 of the possible 130 points scored 49 (or 37,7%). In all four departments, tasks were best done with Pythagorean theorem. Out of a total of 101 points in the first task, all four classes achieved 54 (or 53,5%), while in the second task they achieved 43 or (42,6%). There were less points in the tasks with the use of algebraic formulas. In the third task, the students achieved a total of 32 points (or 31,6%), in the fourth 31 points (or 30,7%) and in the fifth task 26 points (or 25,7%).

Table 1: Pre-test results

Department	In total students	Number of points on the pre-test					Total	Percentage of done tasks
		Task 1	Task 2	Task 3	Task 4	Task 5		
7_1	26	15	12	8	9	7	51	39,2%
7_2	24	12	10	7	7	6	42	35%
7_3	25	13	10	8	7	6	44	35,2%
7_4	26	14	11	9	8	7	49	37,7%

In order to help students to master formulas and expressions better, we organized two additional hours for their presentation and training. Due to the comparison of the results, we formed an experimental and control group of students in which the teaching contents were realized using different teaching materials. Students of the experimental group worked in the computer room, with computers and software package GeoGebra, while the students of the control group worked in the classroom with the help of a board, chalk and pictograms. From classes 7_2 and 7_4, an experimental group (50 pupils) was formed, while the students 7_1 and 7_3 consisted of a control group (51 pupil). In the formation of the experimental and control group we were guided by the principle of approximately the same average grade in mathematics and approximately the same number of students in groups. In order to nurture and achieve cooperative relationships among students in the experimental group, we have formed small three-member collaborative groups from students of different levels of mathematical knowledge (with the exception of two quadruplets). Such groups are recommended by effective learning researchers (Dooly[18], 2008; Chai, Lin, So & Cheah[19], 2011). Collaborative work develops teamwork and responsibility in the work because each individual is responsible not only for his / her own learning but also for learning other members in the group. It also achieves greater efficiency in operation. Students who do not understand the matter ask questions, the students who understood, help them to understand and thus help them, but also deepening and determining their knowledge. In order to achieve good cooperative relationships, students are allowed to group independently with respect to the principles of different levels of mathematical knowledge among group members, because good collaboration is a prerequisite for effective collaborative learning (Dogru & Kalender[20], 2007).

At the beginning of the first class, students in the experimental group were briefly acquainted with the visual way of showing legality and proving the fortification in the ancient, pre-Euclidean period. Then they were told a story about Pythagoras getting acquainted with a right triangle of pages of lengths 3, 4, and 5, which is valid $5^2 = 3^2 + 4^2$, and his conclusion that this lawfulness applies in every right triangle. Then, on the computers, the visual proof of this claim is shown (Figure 1).



1_A 1_B

Figure 1: Pythagorean Theorem

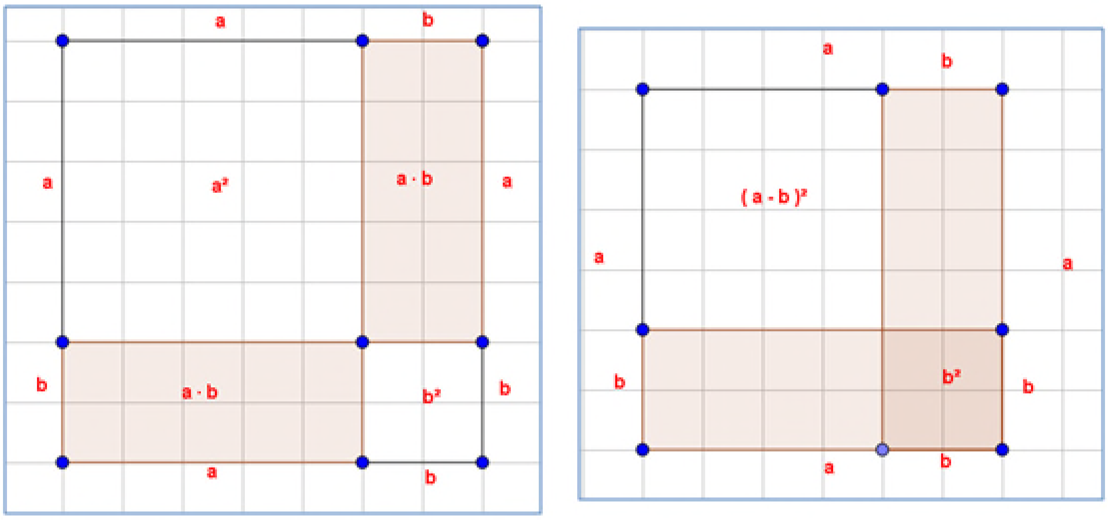
With the visual representation of the students, it was shown that the larger square in Figure 1_A is composed of a smaller square (colored and marked with P_1) and 4 each other equal the right triangle of pages a, b, and c. In Figure 1_B, this larger square is after moving the right triangles of pages a, b, and c composed of these 4 triangles and two smaller squares (colored and marked with P_2 and P_3). The picture clearly shows that $P_1 = P_2 + P_3$. The square marked with P_1 has a length c page, while the squares marked with P_2 and P_3 have pages a and b, respectively. Area of the square $P_1 = c^2$, $P_2 = a^2$ and $P_3 = b^2$. Replacing the obtained expressions in equality $P_1 = P_2 + P_3$, we obtain $c^2 = a^2 + b^2$, thus confirming the Pythagorean theorem.

Then, students were told to use the "Segment" command in GeoGebra to determine the lengths of pages a, b, and c. Lengths 3, 4 and 5 were obtained. Thus, the first Pythagoras triple numbers were obtained at the time (triples of the numbers a, b, c for which $c^2 = a^2 + b^2$ applies).

After that, the students were asked to display a square of page 17 in the GeoGebra and to divide it into 4 straight triangles with catheters in lengths 5 and 12 in the example. It was then required to move the right triangles with a new image with 4 triangles and two squares. It was interesting for students to draw pictures in GeoGebra and all groups successfully completed this task. By applying the "segment" command, the length of the hypotenuse of a drawn straight triangle was obtained (c = 13, the lengths of the cathets were 5 and 12), and in this way they got the other Pythagoras triple numbers 5, 12, 13.

In the same way, another example for the right triangle is shown on pages 6, 8, and 10. After that, tasks with assigned catheters and calculation of hypotenuse, as well as a given hypotenuse and one catheter were calculated and the calculation of the second catheter. For the homework, the students were asked to identify five Pythagoras three numbers.

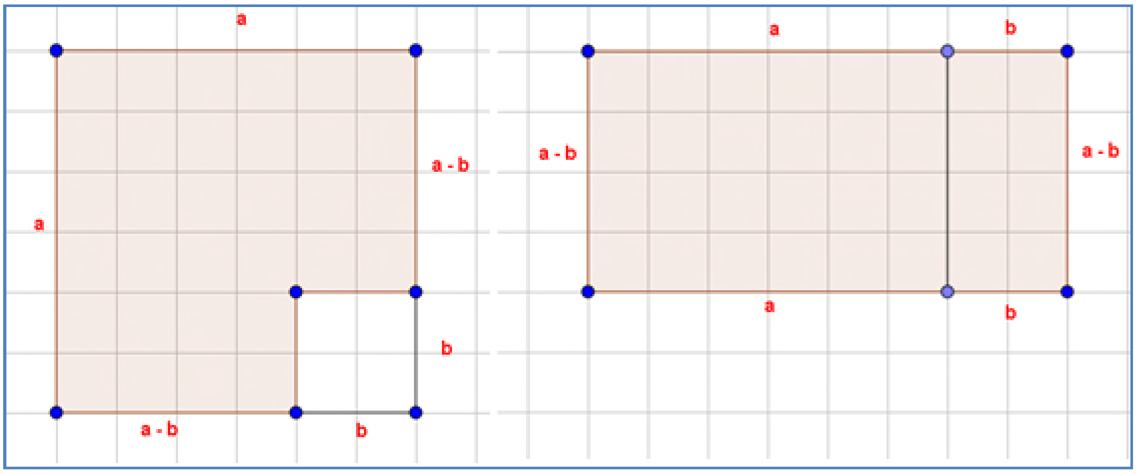
At the beginning of the second time, students were asked what is the equation $(a + b)^2$. The students answered in the choir, there were correct answers, but several answers "$a^2 + b^2$" were heard. Then, on the computers, students are shown in Figure 2_A.



2_A 2_B

Figure 2: Square binomial $(a + b)^2$ and $(a - b)^2$

It is clearly seen in the figure that the large square of the length side a + b contains in itself two smaller squares of sides a and b respectively, but also two rectangles of the lengths a and b. It follows that the surface of a large square is equal to the sum of the surfaces of all its parts, or that $(a + b)^2 = a^2 + 2 \cdot a \cdot b + b^2$. After insight in Figure 2_A, one student's commentary was heard: "Now I understand that $(a + b)^2$ is not equal to $a^2 + b^2$".
Then the students were asked what was the equal $(a - b)^2$. And this time there were the correct answers as well as the answer "$a^2 - b^2$". Figure 2_B is shown on the computers at that time. The picture shows a large square of the page length a, and from which, by subtracting two rectangles, the lengths of the pages a and b (colored in the picture) remain part of the surface $(a - b)^2$. In this case, a small square (painted with a darker color in the picture) with a surface area equal to b^2 was taken twice. From all of the above we see that $(a - b)^2 = a^2 - 2 \cdot a \cdot b + b^2$.
The pupils clearly saw the rules presented in Figure 2, but it was not clear to some what is the same to $a^2 - b^2$. They remember from the previous hours that this term appeared in tasks, but they did not remember what it was the same. Then, on the computers, Figure 3 is shown.



3_A 3_B

Figure 3: Difference squares $a^2 - b^2$

Without further explanation of the teacher, the students saw that the colored surface in Figure 3_A is equal to the surface in Figure 3_B. At the question of the teacher what conclusions are based on the representation in Figure 3, the majority of the groups reached the correct answer: $a^2 - b^2 = (a + b) \cdot (a - b)$. Students who understood the legality shown in Figure 3 helped groups who did not understand explaining that the colored surface in Figure 3_A is equal to $a^2 - b^2$. If the rectangle

on the left side of the small square rotates as shown in Figure 3_B, we get a rectangle with sides of length a + b and a - b whose surface is (a + b) · (a - b), so $a^2 - b^2 = (a + b) \cdot (a - b)$.
After that, tasks with the application of these formulas were made. All groups successfully collaborated with the tasks. Working in the experimental group ended.
Students in the control group over the course of these two hours solved various tasks using the Pythagorean theorem and the aforementioned algebraic formulas. On the wall in the classroom there is an image of a Pythagorean theorem with a right triangle and squares above each page along which the law is written $c^2 = a^2 + b^2$. There were no visual representations for algebraic formulas, but they were derived from the algebraic way and presented to students.
At the third time, both groups were re-tested. Tasks on the test were of the same type and weight as on the pre-test. The results in the experimental group were significantly better than in the control group. After three weeks, we repeated the testing. The difference in post-test performance was even greater then. The results obtained on the test and post-test were presented and analyzed in Section 4.

4. RESULTS AND DISCUSSION

On the test, as well as on the pre-test, there were 5 tasks. Each correct answer is scored with one point and is not a half-scored answer. The results of the test are shown in Table 2.

Table 2: Results of the test

Department	In total students	Number of points on the test					Total	Percentage of done tasks
		Task 1	Task 2	Task 3	Task 4	Task 5		
7_1	26	18	15	11	10	8	62	47,7%
7_2	24	21	17	20	18	21	97	80,8%
7_3	25	15	12	13	10	9	59	47,2%
7_4	26	22	19	22	20	20	103	79,2%

By comparing the results of the pre-test and test, we can see that the students of the control group (7_1 and 7_3) made a progress of 10,2%. At the pre-test they scored a total of 95 points (37,2%) and on the test 121 points (47,4%). The students of the experimental group (7_2 and 7_4) won a total of 91 points (36,4%) on the pre-test, while they scored 200 points (80%) on the test, resulting in a progress of 43,6%. After two additional hours of exercise, both groups made progress, but the progress of the experimental group was considerably higher.
After the test we talked with the students of the experimental group and asked the following questions how they explain their results on the test:
Marko, student 7_2: "As soon as I get the expression with a binomial on a square, I see the square of the page a + b which is not composed only of the square of the page a, and the square of page b already has two pieces of rectangles with sides a and b."
Ana, student 7_4: "I do not know how I used to think that $(a - b)^2$ is equal to $a^2 - b^2$, when they are completely different expressions. And now I clearly see the picture, the result of the first expression is a square, smaller than the initial one, and the result of the second expression is the square from which the smaller square is drawn. "
There were the other interesting answers, and everyone basically had a visual experience of algebraic formulas.
After three weeks, we repeated the testing. Post-test tasks were the same weight and with the same requirements as the test and pre-test. The post-test results are shown in Table 3.

Table 3: Post-test results

Department	In total students	Number of points on the post-test					Total	Percentage of done tasks
		Task 1	Task 2	Task 3	Task 4	Task 5		
7_1	26	16	14	10	8	8	56	43,1%
7_2	24	19	15	20	17	21	92	76,7%
7_3	25	14	10	10	6	7	47	37,6%
7_4	26	22	17	21	18	19	97	74,6 %

Analyzing the results of the pre-test, test and post-test of the control and experimental group, we note that the percentage of accurately performed tasks in the control group was in the order of 37,2%, 47,4%, 40,35%, while in the experimental group it was 36,4 %, 80% and 75,6%. The experimental group achieved better results from the control group by 35,25% at the post-test. The difference in the achievement of the groups is more noticeable and obvious in the graphic representation of these results (Figure 4).

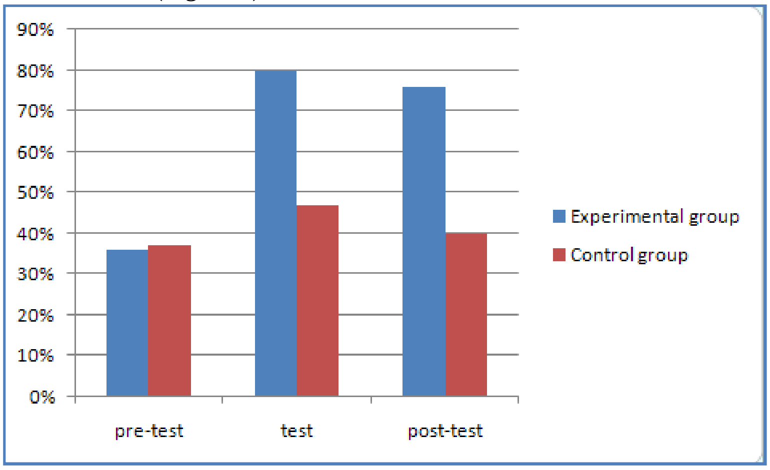


Figure 4: Percentage of exactly done tasks

5. CONCLUSION

Pictorial representation of the legitimacy expressed in mathematical formulas on computers via GeoGebra gave excellent results. Students have seen the work presented as learning through the game. In conversation with them, we learned that the visualization of mathematical principles expressed in formulas made these formulas clear, obvious, and therefore easy to understand and remember. We can conclude that the visualization of mathematical laws is in a positive correlation with the understanding and the memory of the formulas that express the observed lawfulness.
In this paper we have shown that visual representation of a certain legality contributes to understanding this legality and facilitates its memorization.We have shown that collaborative work in groups gives good results. Students who were quicker to master the formulas explained to the other members of the group that contributed to their faster progression.
We have also shown that the use of modern technologies produces multiple benefits in teaching. Learning makes it easier and more interesting and enables students to use modern technologies for cognitive purposes. Thus, one of the main goals in education is fulfilled - training of students for active and purposeful use of computers and software packages intended for education.
Based on the above, we can conclude that mastering mathematical formulas through computers and GeoGebra is very effective and efficient. Learning in the GeoGebra environment, students could get the length of their pages with the image of a straight triangle, thus checking their knowledge as well as the accuracy of the done task. The algebraic formulas for the square of the binomial and the difference of squares, having obtained their image presentation, have become visually clear and therefore easy to understand and remember. We can also conclude that learning in collaborative groups is much more efficient than individual learning. Students in collaborative work have a sense of responsibility for the success of the entire group, which contributes to greater motivation in work and better results.

LITERATURE

[1] Duval, R. (1999). Representation, Vision and Visualization: Cognitive Functions in Mathematical Thinking. Basic Issues for Learning.
[2] Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational studies in mathematics*, *52*(3), 215-241.
[3] Scheiter, K., Gerjets, P., & Schuh, J. (2010). The acquisition of problem-solving skills in mathematics: How animations can aid understanding of structural problem features and solution procedures. *Instructional Science*, *38*(5), 487-502.
[4] Rivera, F. D., & Becker, J. R. (2008). Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. *ZDM*, *40*(1), 65-82.
[5] Van Garderen, D., & Montague, M. (2003). Visual-spatial representation, mathematical problem solving, and students of varying abilities. *Learning Disabilities Research & Practice*, *18*(4), 246-254.

[6] Pape, S.J., Bell,C.V.&Yetkin,I.E.(2003).Developing mathematical thinking and self-regulated learning:A teaching experiment in a seventh-grade mathematics classroom. *Educational Studies in Mathematics*, *53*(3), 179-202.
[7] Gomez, E. A., Wu, D., Passerini, K. (2010).Computer-supported team-based learning: the impact of motiva tion, enjoyment and team contributions on learning outcomes. *Computers & Education*, 378-390.
[8] Garcia, C. P. (2013). A constructivist computational platform to support mathematics education in elementary school. *Computers & Education*, 66, 25-39.
[9] Ruthven, K. (2009). Towards a naturalistic conceptualisation of technology integration in classroom practice:The example of school mathematics. *Education and Didactique*,3(1),131–159.
[10] Manĕnová, M., Skutil, M., & Zikl, P. A. V. E. L. (2010). Taking advantage of ITC by teachers at the primary school. In *Proc. of the 6th Educational technologies (EDUTE'10): WSEAS/IASME international conference. Athens: WSEAS* (pp. 48-52).
[11] Valtonen, T., Havu-Nuutinen, S., Dillon, P., Kontkanen, S., Vesisenaho, M., Pontinen, S. (2014). Challenges with social software for collaboration: two case studies from teacher training. *International Journal of Media, Technology & Lifelong Learning*, 10(1).
[12] Viamonte, A. J. (2010). The computer in the teaching of mathematics. In Proceeding of advanced educational technologies. 6th WSEAS/IASME international conference on educational technology (EDUTE'10) (pp. 24-29).
[13] Takači, Đ., Stankov, G., Milanovic, I. (2015). Efficiency of learning environment using GeoGebra when calculus contents are learned in collaborative groups. *Computers and Education*, 82, 421-431.
[14] Ruthven, K., Hennessy, S., Deaney, R. (2008). Constructions of dynamic geometry: A study of the interpretative flexibility of educational software in classroom practice. *Computers and Education*, 51(1), 297–331.
[15] Laborde, C. (2001). Integration of technology in the design of geometry tasks with Cabrigeometry. *International Journal of Computers for Mathematical Learning*, 6, 283– 317.
[16] Lavicza, Z., Papp-Varga, Z.(2010). Integrating GeoGebra into IWB-equipped teaching environments: preliminary results. *Technology, Pedagogy and Education*, 2, 245-252.
[17] Walny, J., Carpendale, S., Riche, N. H., Venolia, G. & Fawcett, P. (2011). Visual thinking in action:Vizualizations as used on whiteboards. *IEEE Transactions on Visualization and Computer Graphics*, *17*(12), 2508-2517.
[18] Dooly, M. (2008). Constructing knowledge together. In M. Dooly (Ed.), *Telecollaborative language learning. A guidebook to moderating intercultural collaboration online* (pp.21- 24). Bern: Peter Lang.
[19] Chai, C., Lin, W.-Y., So, H.-J., Cheah, H.M. (2011). *Advancing collaborative learning with ICT: Conception, cases and design.* Singapore: Ministry of Education.
[20] Dogru, Kalender, (2007). Applying the subject 'cell' through consructivist approach during science lessons and the teacher's view. *Journal of Environmental & Science Education*, 2(1), 3-13.