

ON LIMITNESS OF KNOWLEDGE FROM THE PERSPECTIVE OF MATHEMATICS

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Abstract: *In this paper, authors present several known mathematical fields and their characteristic results, which show that every field is on some way limited – is it our human limit in thinking, or is it an objective limitness of mathematics that we know nowadays? Its place in this paper will find Godel's incompleteness theorem, Zermelo's axiom of choice, some problems in graph theory, probability, number theory, geometry and Ramsey's theory.*

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1. INTRODUCTION

This article is author's attempt to present mathematics as a scientific field with no great differences than other natural sciences, if we talk about unperfectness. We cannot, hence, talk about mathematics as a perfect human creation, which was developed solely on the basis of great ideas of great people, with the flame and sparks that burnt in their soles. In that manner, authors will try to reach some dose of originality.

Knowing the fact that, in mathematics, we can freely set axioms and choose whether to accept or decline some statement to be axiom, it led to big changes in fundamentals of mathematics. It took only a few decades to completely abandon the traditional standing and to come to new conclusions and a new believing - that every mathematical thinking can be represented as formal system, in which axioms and theorems are only sequences of symbols and nothing more. Well known philosopher, Bertrand Russell, gives us a bit paradoxical conclusion on this theme: „Mathematics can be defined as a field in which nor do we know what we are talking about, nor do we know if that what we've said is true.“

2. MATHEMATICAL FORMAL SYSTEMS; AXIOM OF CHOICE

2.1. Mathematics as formal system

By accepting Russell's claim that mathematical concepts don't have to have actual meaning, we can conclude that whole mathematics can be reduced to set of symbols and rules for their manipulation. This concept, however, has some obvious advantages.

It is proven that every system of that kind has to contain only three types of rules:

- rules of formation, which determine whether some sequence of symbols belongs to system or not;
 - axioms, i.e. „truths“ that are accepted as truths without any kind of proof (base); and
 - rules of reasoning, which precise the way on which is some theorem derived (proven) with theorems we already have.
- Let's show one example. If we assume that we have formal system which contains three kinds of symbols: letters **P**, **Q** and sequences of dashes (which can be of the various length). Rules and axioms that define this system can be described on the way below.

Rules of formation. Every set of symbols which is in form

$$xPyQz \tag{1}$$

belongs to the system, where x , y and z are sequences of dashes. We say that this system is correctly formed.

Axioms. Axioms are every set of symbols in the form

$$xP - Qx \tag{2}$$

where x is as sequence of dashes.

Rules of reasoning. If a sequence of symbols

$$xPyQz \tag{3}$$

is theorem, then also a sequence

$$xPy - Qz - \tag{4}$$

is theorem. Notice that in this system certain sequences of symbols are not allowed. On example, sequence

$$- - - P - - Q - - - \tag{5}$$

is correctly formed, but sequences

$$- - - P Q - - - \tag{6}$$

and

$$- P - Q - P \tag{7}$$

are not, because they don't satisfy rules of formation. Let's also highlight that theorems of PQ system can be derived from axioms of that system, by directly applying rules of reasoning.

If we, say, go from axiom

$$- - P - - - \tag{8}$$

theorem

$$- - P - - Q - - - - \tag{9}$$

follows automatically; if we apply reasoning rule once more, we get

$$- - P - - - Q - - - - - \tag{10}$$

etc. In fact, it would be very easy to write a computer program that can generate theorems on described way. That program would never end, because number of possible theorems is unlimited.

2.2. Zermelo's axiom of choice

It is not impossible that Russel pronounced his famous opinion about mathematics, because he was under the influence of Zermelo's axiom of choice. Let's recall this important axiom:

Axiom: „If S is a given set, than from every unempty subset of S one element can be chosen i.e. there is at least one function that picks one element from every unempty subset $X \hat{=} S$.”

Every function that satisfies this condition, is called function of choice in the set S . In other words, function of choice in S is every mapping

$$f : P(S) \setminus \{\emptyset\} \rightarrow S \tag{11}$$

with the property that $f(x) \in X$.

That axiom is one of the hardest and most discussionable mathematical questions. Axiom is obviously correct for limited and countable sets, but when we try to apply it on uncountable sets, we have a great problem because axiom doesn't give us an effective method for element choosing, it only claims that such mapping exists. Some mathematicians, hence, accept axiom of choice, and others don't.

Mathematician Bertrand Russel once told about axiom of choice this: „In the beginning, axiom looks very obvious, but as more as you dive into it, you find consequences of it to be more and more strange. Once you stop understanding what it really represents.“

On this place, it would be natural to also cite Russel's paradox: „Let S be a set of all sets that don't contain them as their element: $S = \{x \mid x \notin S\}$. A question arises: does set S belong to itself i.e. $S \in S$?“

If S is not an element of S , then it is, according to definition of set S , $S \in S$. Otherwise, if $S \in S$, then, according to definition, we have that $S \notin S$. In both cases we came to paradox.

2.3. Sets not measurable according to Lebeg

Is every set A on circle measurable according to Lebeg i.e. could we find measure $m(A)$ of that set? It has been proved that the answer is negative.

Naimely, Vitaly is the first mathematician that gave an example of set that is not measurable, by using axiom of choice. However, we cannot completely throw away this axiom, because it is used to prove equivalence of two definitions of function border value (limes) in mathematical analysis.

Otherwise, how important Lebeg's theory of measure is, we can see from the fact that mathematical theory of probability couldn't be formed until this measure theory was introduced (russian mathematician Kolmogorov then told that *mathematical theory of probability was on the palm*).

2.4. Gedel's First theorem on incompleteness; algebra's fundamental theorem

During the first few decades of 20th century, there was an opinion that whole mathematic knowledge can be reduced to a set of formal rules and symbols, as we mentioned in Introduction, particularly after the publication of book „Principles of mathematics“ written by Bertrand Russel and Alfred North Whitehead. Many of mathematicians hoped that finally all of inconsistencies and doubts are removed from mathematics and that manhood got only a few steps from Platon's vision of „ideal human knowledge“.

However, that first rush of optimism was brutally stopped by austrian mathematician Kurt Gedel: in his paper „*On formal indeterminacy of statements in Principles of mathematics and related systems*“ (1931).

So what did Gedel actually prove? His basic result can be defined by following theorem:

(*Gedel's First theorem on incompleteness*) „*In every system that is complex enough, there are undetermined statements.*“

Even if we try to add infinite number of axioms and rules, there will always be at least one true statement about natural numbers, that can not be proved in that formal system. (We will, of course, not try to prove this theorem in this small paper).

This theorem led Gedel to publish his further statement: „*If mathematics really describes objective reality, I don't see why in it we could not apply inductive methods that are used in physics.*“

Let's also see what is happening with algebraic proof of Algebra's fundamental theorem. Recall the statement of that theorem: „*If $f(x)$ is a polynomial with real coefficients, it has to have at least one zero in the field of complex numbers.*“ We can represent this theorem's statement on the other way: „*A field of complex numbers is algebraically closed.*“

There are hundreds of proofs of this theorem, but none of them uses only apparatus of algebra – these proofs are not algebraic in nature. The question arises: is it possible to find solely algebraic proof of Algebra's fundamental theorem, or it is one of the statements that are unprovable in algebra or universal algebra (Gedel's theorem)?

So, for those who wanted safety and absolute precision, Gedel's theorem was ultimately a bad surprise.

We will cite one more Bertrand Russel's comment: „*I wanted safety, like people that join religions have, and I considered that I will find it in mathematics. Soonly, I realized that proofs that my professors presented are full of errors and that safety in mathematics could only be achieved by enforcing its logical foundations. [...] After 20 years of hard work on this problemacy, I've concluded that nothing more can be done.*“

Of course, that doesn't mean that mathematics lost its importance after Gedel. Physicist Freeman Dyson says: „*When mathematicians recovered from the initial shock, they had realized that Gedel's theorem is actually a good stuff for their profession. The fact that there is no algorithm that can precisely answer on all our questions, is interpreted as a sign that mathematics will never become superfluous.*“

Imperfection of mathematics arises one more interesting question. Is we cannot be absolutely sure in correctness of our results, according to which criterias we can accept some statement as true or false?

In that manner, let's cite one example that points on possible problems. It is famous Ferma's great theorem (known also as Ferma's second theorem or hypothesis) which claims that there are no natural numbers x , y and z that satisfy the equation

$$x^n + y^n = z^n \tag{12}$$

where n is an integer greater than 2. This hypothesis was made in 1632., and in the next 350 years no one could prove that it is correct, or could find some counterexample showing that theorem is not correct.

In 1995., mathematician Andrew Wales finally published proof of this theorem. That proof is so complex and long (it has few hundred pages) that only a few mathematicians actually completely understood it. This proof became widely accepted in world of mathematics, besides almost no one actually tried to check it in details.

Johnatan Borwine says about it: „*Maybe 50 of 100 people nowadays understand Wales’s proof of Ferma’s great theorem (with enough effort and time). If probability that everyone of them overseen one mistake in the proof is just 1%, it leads to the conclusion that many of numerical results are far more reliable than the proof of Ferma’s great theorem.*“

3. PROBLEMS IN MATHEMATICAL ANALYSIS

Many mathematicians od 18th and 19th century used term of simply closed curve rather instinctively than formally. That principle was kept until french mathematician Jordan gave the next definition of simply closed curve: he defined it as a continuous and bijective mapping of circle.

Because curcle separates space (more precisely – plane in which it is located) on two distinct areas, one limited and one unlimited, it looks very logical to assume that mentioned bijective and continuous mapping will have the same properties. However, it is not the case: continuous and bijective mapping of circle can look pretty „wild“. It can, on example, have an infinite length, or it can (even worse) be a curve without any tangenta!

First recorded example of function of that kind gave us Weierstrass, who concluded that next mapping:

$$\begin{cases} x = \sin(Q); \\ y = \frac{1}{2^n} \cos(3^n Q) \end{cases} \quad (13)$$

is continuous and bijective (obviously true for x ; for y we have one geometrical array, $\frac{1}{2^n}$, whose limit value is 0, and value of $\cos(3^n Q)$ belongs to segment $[-1,1]$). In that way, if angle Θ continuously picks values from 0 to 2π , point

$$(x(\Theta), y(\Theta)) \quad (14)$$

continuously slides on some curve. If that curve has tangenta is som of its points, it has to have derivations

$$dx/d\Theta; dy/d\Theta. \quad (15)$$

But, when we actually try to find derivation of y , it is easily provable that it doesn’t have first derivation for any angle Θ ! So, it is obvious that Jordan’s definition of continuous curve is not good enough because it also defines mappings that are not continuous, and also those that are “not normal” in terms of our natural intuition.

4. RAMSEY’S THEOREM

In 1930, english mathematician and logician Franck P. Ramsey had proved theorem, whose special case is Dirichet’s principle. His researches in “formal logic” and discovered theorem, much later found their places in combinatorics, namely in graph theory. As we have at Dirichlet’s principle, also here it is claimed that certain numbers exist (in Dirichlet’s principle – at least two numbers in “strong form”

$$\frac{\hat{c}n - 1}{\hat{c}n} \frac{\hat{u} + 1}{\hat{u}} \quad (16)$$

Ramsey’s theorem ensures existance of certain numbers with so called Ramsey’t properties. Effective values of these numbers are very hard to find and their properties are not yet completely solved. Nowadays, only few of Ramsey’s numbers are known.

One typical problem will give us a clearer picture of what Ramsey’t theory is addressing.

„*Let n be an integer. Which is the minimal number of people on one, say, meeting, that it would be certain that there is a subset of k people in which all know each other, or k people that no one of them knows remaining $k-1$ people?*“

If we denote that number with r , Ramsey’s theorem gives us the proof that r really exists, but we don’t get any effective method for finding such value. Today, we know only few of r values, and only for specific values of k .

So, what is the basic problem for finding these Ramsey’s numbers, remains a mistery. Maybe it is really similar to Ferma’s great theorem?

5. CONCLUSION

In this short paper, we had to limit the choice of touched themes and methods of modern theoretical and applied mathematics. That choice, of course, carries a dose of subjective taste of authors. Some modern methods and techniques we could not mention.

Our goal was to constantly remain on trace of unsolved mathematical problems, fundamental discoveries and interesting things that, without asking us, lead us to conclusion. Even if mathematics is sometimes considered as a queen of sciences, because it has its applications in almost every other science, we have to accept the fact that mathematics has, just like other sciences, its limited reaches. That kind of conclusion surely leads us to the way on which we cannot lose the measure of objectiveness.

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